数理解析研究所講究録
第 867 巻 1994 年 73-87

On Geometries of Type C_3 and their Extensions

吉荒 聡 Satoshi Yoshiara
Division of Mathematical Sciences
Osaka Kyoiku University
Kashiwara, Osaka 582, JAPAN

Abstract

This is an extended and updated version of my earlier talks given in [Yo2] and [Yo3]. After describing motivations of the investigation at some length, recent progress in classification of flag-transitive C_3- and $C_3.c^*$-geometries is reported.

1. Fundamental Definitions and Examples.

As is always, I will begin by recalling some fundamental terminologies.

1.1 Notation. An *incidence geometry* over an ordered set $I = \{0, \ldots, r-1\}$ is a multipartite graph $\mathcal{G} = (\mathcal{G}_0, \ldots, \mathcal{G}_{r-1})$ with (ordered) parts $\mathcal{G}_i$ indexed by I, in which each clique (usually called a *flag*) is contained in a maximal clique of size r. For each flag X, the subset $type(X) := \{i \in I | \mathcal{G}_i \cap X \neq \emptyset\}$ of I is the *type* of X and its complement $I - type(X)$ is the *cotype* of X.

The cardinal r of I is the *rank* of the geometry $\mathcal{G}$. We usually use the term *varieties* to refer to vertices of $\mathcal{G}$, and two varieties are called *incident* if they are adjacent or coincide. The varieties in $\mathcal{G}_0$ (resp. $\mathcal{G}_1$ and $\mathcal{G}_2$) are usually called *points* (resp. *lines* and *planes*).

Two geometries $\mathcal{G}$ and $\mathcal{H}$ over the same ordered set I are called *isomorphic* if there is a bijective map f from $\mathcal{G}$ to $\mathcal{H}$ sending $\mathcal{G}_i$ to $\mathcal{H}_i$ for each $i \in I$ such that two varieties x, y of $\mathcal{G}$ are incident in $\mathcal{G}$ iff $f(x)$ and $f(y)$ are incident in $\mathcal{H}$. Two geometries $\mathcal{G}$ and $\mathcal{H}$ of rank 2 over the ordered set $I = \{0, 1\}$ are called *dual* if there is a bijective map f from $\mathcal{G}$ to $\mathcal{H}$ sending $\mathcal{G}_0$ (resp. $\mathcal{G}_1$) to $\mathcal{H}_1$ (resp. $\mathcal{H}_0$) such that two varieties x, y of $\mathcal{G}$ are incident in $\mathcal{G}$ iff $f(x)$ and $f(y)$ are incident in $\mathcal{H}$.

For a flag X of cotype J and an index $j \in J$, we write $\mathcal{G}_j(X) := \{y \in \mathcal{G}_j | \{y, X\} \text{ is a flag }\}$. The subgraph of $\mathcal{G}$ induced on the set of varieties incident to every varieties in X but not in X is a multi-partite graph with parts $\mathcal{G}_j(X)$ $(j \in J)$ indexed by J, and so it can be thought of as an incidence geometry on J. This is called the *residue* of (or at) X in (or of) $\mathcal{G}$, and denoted by $Res_{\mathcal{G}}(X)$ (or $Res(X)$ for short when $\mathcal{G}$ is well understood). The cardinal $|type(X)|$ is the *corank* of the residue $Res(X)$, and so $Res(X)$ is a geometry of rank $|I|-$the corank of $Res(X)$.

Note that the ordering of J is inherited from that of I. Thus, for example, if we take a flag X of cotype $\{i, j\}$ with $i < j$, the residue $Res(X)$ is a multipartite graph $(\mathcal{G}_i(X), \mathcal{G}_j(X))$ over $\{i, j\}$, which is *not*, in general, isomorphic to the geometry $(\mathcal{G}_j(X), \mathcal{G}_i(X))$ over $\{j, i\}$, the dual of $Res(X)$.

If there exists a constant number s_i such that there are exactly $s_i + 1$ maximal flags containing each flag of cotype $\{i\}$, this number s_i is called the i-th *order* of a geometry $\mathcal{G}$. A geometry $\mathcal{G}$ over I is said to ***have orders*** if s_i exist for all $i \in I$. In this case, $(s_0, \ldots, s_{r-1})$ is called the *order* of $\mathcal{G}$. If all orders are finite, $\mathcal{G}$ is said to be *locally finite.* A geometry $\mathcal{G}$ is called *thick* (resp. *thin*) if there are at least three (resp. exactly two) maximal flags containing each flag of corank 1.

The isomorphisms from a geometry $\mathcal{G}$ to itself form a group with respect to the composition of maps, which is denoted by $Aut(\mathcal{G})$ and called the (special) *automorphism group* of $\mathcal{G}$. If there is a homomorphism ρ from a group G to $Aut(\mathcal{G})$, we say that G ***acts on*** $\mathcal{G}$ (or $\mathcal{G}$ *admits* G) and the kernel of ρ is called the *kernel* of the action. If a group G acts on $\mathcal{G}$, we denote by G_X the *stabilizer* of a flag X, that is, the subgroup of G of elements stabilizing X globally. Since isomorphisms of $\mathcal{G}$ preserve each part $\mathcal{G}_i$, G_X acts on the geometry $Res(X)$. The kernel of this action is denoted by K_X. That is, K_X is the normal subgroup of G_X fixing each variety contained in X, and hence G_X/K_X is isomorphic to a subgroup of $Aut(Res(X))$.

A group G is called *flag-transitive* on $\mathcal{G}$ if G acts transitively on the set of maximal flags. A geometry $\mathcal{G}$ is *flag-transitive* if it admits a flag-transitive group. If G is flag-transitive then the stabilizer G_X is flag-transitive on $Res(X)$ and so G_X/K_X is a flag-transitive subgroup of $Aut(Res(X))$. Furthermore, if $\mathcal{G}$ is flag-transitive, $\mathcal{G}$ has orders.

Now I will give standard examples of incidence geometries, some of which will be analized later.

1.2 Example (Projective spaces). Let V be a (right) vector space over a division ring K of dimension $r+1$. Defining $\mathcal{G}_i$ as the set of $i+1$-dimensional subspaces of V $(i = 0, \ldots, r-1)$ and incidence by (symmetrized) natural inclusion, we have a geometry $\mathcal{G}$ of rank r. This is called the *projective space* associated with V, and denoted by $\mathbf{PG}(V)$. The automorphism group $Aut(\mathbf{PG}(V))$ is an extension of $PGL(V)$ by the group of field automorphisms, which is flag-transitive on $\mathbf{PG(V)}$. The order of $\mathbf{PG}(V)$ is $(|K|, |K|, \ldots, |K|)$.

1.3 Example (Finite classical polar spaces). Let (V, f) be one of the following pairs of a vector space over a finite field and a form on it.

(W_{n-1}) V is a vector space of dimension $n = 2r$ over $GF(q)$ equipped with a non-degenerate symplectic form f (of Witt index r).

(H_{n-1}) V is a vector space of dimension n over $GF(q^2)$ equipped with a non-degenerate hermitian form f (of Witt index $r = [n/2]$).

(Q_{n-1}) V is a vector space of dimension $n = 2r + 1$ over $GF(q)$ equipped with a non-singular quadratic form f (of Witt index r).

(Q^+_{n-1}) V is a vector space of dimension $n = 2r$ over $GF(q)$ equipped with a non-singular quadratic form f of Witt index r.

(Q^-_{n-1}) V is a vector space of dimension $n = 2(r+1)$ over $GF(q)$ equipped with a non-singular quadratic form f of Witt index r.

For $i = 0, \ldots, r-1$, we define $\mathcal{G}_i$ to be the totally isotropic (or singular) subspaces of dimension $i+1$ with respect to the form f. By defining the incidence by inclusion, we have a geometry $\mathcal{G} = W_{2r-1}(q) = (\mathcal{G}_0, \ldots, \mathcal{G}_{r-1})$ of rank r if (V, f) is of type W_{2r-1}, which is called a *symplectic* polar space. Similarly, if (V, f) is of type (H_{n-1}), setting $[n/2] = r$, we have a geometry $\mathcal{G} = H_{2r-1}(q^2) = (\mathcal{G}_0, \ldots, \mathcal{G}_{r-1})$ of rank r, called a *hermitian* polar space. For (V, f) of type (Q_{2r}), we have a geometry $\mathcal{G} = Q_{2r}(q) = (\mathcal{G}_0, \ldots, \mathcal{G}_{r-1})$ of rank r, called a *neutral* polar space. For (V, f) of type (Q^+_{2r-1}) (resp. (Q^-_{2r+1})), we have a geometry $\mathcal{G} = Q^+_{2r-1}(q) = (\mathcal{G}_0, \ldots, \mathcal{G}_{r-1})$ of rank r (resp. $\mathcal{G} = Q^-_{2r-1}(q) = (\mathcal{G}_0, \ldots, \mathcal{G}_{r-1})$ of rank r), called a *hyperbolic* polar space (resp. an *elliptic* polar space).

These five families of (finite) polar spaces are called *classical* polar spaces. Note that the hyperbolic polar spaces Q^+_{2r-1} are not thick, because there are exactly two totally singular subspaces of dimension r containing each totally singular subspace of dimension $r-1$. We can verify that the order of $W_{2r-1}(q)$ (resp. $H_{2r-1}(q^2)$, $H_{2r}(q^2)$, $Q_{2r}(q)$, Q^+_{2r-1}, and Q^-_{2r-1}) is $(q, \ldots, q, q)$ (resp. $(q^2, \ldots, q^2, q)$, $(q^2, \ldots, q^2, q^3)$, $(q, \ldots, q, q)$,$(q, \ldots, q, 1)$, and $(q, \ldots, q, q^2)$). The automorphism group of each polar space is the projective semi-linear classical groups associated with (V, f) (that is, the groups of non-singular linear transformations on V projectively preserving the form f extended by the field automorphisms (if they exist)), which acts flag-transitively on the polar space.

1.4 Buildings. The above examples 1.2, 1.3 (and also 2.2 and 2.3 below) belong to an important class of geometry, called *buildings.* I omit to give a formal definition of buildings, but you may consult [Ro] Chap.3 and [Ti1] Chap. 1–3).

It is shown by Tits [Ti1] that thick buildings of rank $r \geq 3$ and of "spherical type" should be one of these classical geometries such as projective spaces in Example 1.2 (buildings of type A), classical polar spaces in Example 1.3 (with some modification using sesquilinear forms in the infinite case and some other geometries in the case of rank 3) (buildings of type $B = C$) and the geometries associated with hyperbolic polar spaces (buildings of type D) as well as those related to exceptional simple algebraic groups of type F_4, E_6, E_7 or E_8.

1.5 Example (A tower of classical extended polar spaces). Let Ω be a set of $2n+2$ letters with $n \geq 2$, and define $\mathcal{G}_i$ to be the family of subsets of Ω consisting of $2(i+1)$ letters ($i = 0, \ldots, n-2$). We define $\mathcal{G}_{n-1}$ to be the set of all partitions of type 2^{n+1} of Ω. Incidence is given by inclusion among varieties of $\cup_{i=0}^{n-1}\mathcal{G}_i$ and a subset $T \in \cup_{i=0}^{n-1}\mathcal{G}_i$ is incident to a partition $\{T_1, \ldots, T_{n+1}\} \in \mathcal{G}_{n-1}$ whenever T is a union of some T_i and T_j. The resulting geometry is denoted by $\mathcal{S}_{2n+2}$.

2. Some geometries of rank 2.

Now let me recall some families of geometries of rank 2, importance of which will be explained in §3. First I introduce a geometry consisting vertices and edges of some (interesting) graphs.

2.1 Definition (Vertex-Edge Geometry of a Graph). Let $\Gamma = (V, E)$ be a graph with the sets V and E of vertices and edges, respectively. The geometry $\mathcal{G}(\Gamma) = \mathcal{G} = (\mathcal{G}_0, \mathcal{G}_1)$ over $\{0, 1\}$ with $\mathcal{G}_0 = V$, $\mathcal{G}_1 = E$ and incidence given by natural inclusion is called the *vertex-edge* geometry of Γ. Clearly $Aut(\mathcal{G}(\Gamma))$ coincides with the full automorphism group of the graph Γ. There are exactly two "points" incident to each "line" of $\mathcal{G}$, and if Γ is a graph of valency k, there are exactly k "lines" incident to each "point" of $\mathcal{G}$. Thus, the geometry $\mathcal{G}(\Gamma)$ for a regular graph Γ of valency k has order $(1, k-1)$.

The point-edge graph of complete graphs and the Petersen graph are turned out to be very important and now called *circle geometry* and the *Petersen geometry.* We denote by $\mathcal{C}_n$ the circle geometry with n "points" (and so $n(n-1)/2$ "lines") and by $\mathcal{P}$ the Petersen geometry (with 10 "points" and 15 "lines"). The circle geometry $\mathcal{C}_n$ has order $(1, n-2)$ and $Aut(\mathcal{C}_n) \cong S_n$ (the symmetric group of degree n). The Petersen geometry $\mathcal{P}$ has order $(1, 2)$ and $Aut(\mathcal{P}) \cong S_5$.

The next example is the most important family of geometries.

2.2 Definition (Generalized Polygons). Let n be a natural number. A *generalized n-gon* is a geometry $\mathcal{G} = (\mathcal{G}_0, \mathcal{G}_1)$ of rank 2 with diameter n and girth $2n$, such that for every vertex x there is a vertex y at distance n from x. The varieties in $\mathcal{G}_0$ and $\mathcal{G}_1$ are called *points* and *lines* respectively.

In the above, the distance between two varieties of $\mathcal{G}$ are defined to be the length of a shortest path joining them, the diameter is the maximum distance between two varieties of $\mathcal{G}$, and the girth is the smallest number of varieties appearing in a circuit (without backtrack). In particular, $\mathcal{G}$ is a connected bipartite graph, and hence $n \geq 2$.

2.2 Generalized polygons with orders. It is not difficult to verify that a generalized n-gon has orders (s, t) (that is, there are constant numbers s and t with $s, t \geq 2$ such that there are exactly $s + 1$ points (resp. $t + 1$ lines) incident to each line (resp. point), where s and t are possibly infinite) if it is thick (that is, every variety x of a generalized n-gon $\mathcal{G}$ is incident to at least three varieties different from x). By the remarkable theorem of Feit and Higman, if a generalized n-gon $\mathcal{G}$ has order (s, t) with $s \geq 2$ and $t \geq 2$ then either $\mathcal{G}$ has an infinite number of varieties or $n = 2, 3, 4, 6$ or 8.

On the other hand, even if we restrict our interests to finite generalized n-gons (that is, those with finite number of varieties) having order (s, t), we cannot restrict n if $s = 1$ or $t = 1$. An example of a generalized $2n$-gon $\mathcal{F}$ of order $(1, t)$ can be easily constructed as follows starting from a generalized n-gon $\mathcal{G}$ of order (t, t): Let $\mathcal{G}$ be a generalized n-gon of order (s, t) in general. Define the POINTS of $\mathcal{F}$ as the varieties of $\mathcal{G}$ and define the LINES of $\mathcal{F}$ as the maximal flags of $\mathcal{G}$. Incidence is given by the natural inclusion. The resulting graph $\mathcal{F}$ is the subdivision graph of $\mathcal{G}$ (that is, the graph obtained from $\mathcal{G}$ by recognizing edges of $\mathcal{G}$ as new vertices), and hence we can easily verify that $\mathcal{F}$ is a generalized $2n$-gon in which there are exactly two POINTS incident to a LINE (that is, $s = 1$) and there are exactly $(s + 1)$ (resp. $(t + 1)$) LINES incident to a POINT if it corresponds to a point (resp. line) in $\mathcal{G}$. Thus if $\mathcal{G}$ has order (t, t), the generalized $2n$-gon $\mathcal{F}$ has order $(1, t)$.

2.3 Geometric interpretations. For $n = 2, 3, 4$, the generalized n-gons has the following geometric characterizations. Let $\mathcal{G} = (\mathcal{P}, \mathcal{L}; *)$ be a geometry of rank 2.

(1) $\mathcal{G}$ is a generalized 2-gon if and only if $\Gamma(\mathcal{G})$ is a complete bipartite graph if and only if $P * l$ for every points P and every lines l.

(2) $\mathcal{G}$ is a generalized 3-gon if and only if $\mathcal{G}$ is a (generalized) projective plane: that is, there is a unique point (resp. line) incident with given two distinct lines (resp. points).

(3) $\mathcal{G}$ is a generalized 4-gon if and only if there is at most one line (resp. point) incident with two distinct points (resp. lines) and for any point P and an line l not incident with P there exist a point Q incident with l and a line m incident with Q and P.

2.4 Example (Desarguesian projective planes). Most popular explicit examples of generalized 3-gons is the Desarguesian projective plane $\mathbf{PG}(2, q)$ associated with the 3-dimensional vector space $GF(q)^3$ over the finite field $GF(q)$. This is the rank 2 case of the example 1.1.

2.5 Example (Classical GQs). As for a generalized 4-gon, which we will call a *generalized quadrangle* and abbreviate to GQ, the classical polar spaces in Example 1.2 give examples if they are of rank 2. They are called the *classical GQ*. Explicitly, they consist of the following five families, where $q = p^e$, p a prime:

$W_3(q)$ The symplectic GQ $W_3(q)$ of order (q, q) admitting the flag-transitive automorphism group isomorphic to the projective symplectic group $PGSp_4(q)$ extended by the field automorphism (of order e).

$H_3(q^2)$ The Hermitian GQ $H_3(q^2)$ of order (q^2, q) admitting the flag-transitive automorphism group isomorphic to the projective unitary group $PGU_4(q^2)$ extended by the field automorphism of order $2e$.

$H_4(q^2)$ The Hermitian GQ $H_4(q^2)$ of order (q^2, q) admitting the flag-transitive automorphism group isomorphic to the projective unitary group $PGU_5(q^2)$ extended by the field automorphism of order $2e$.

$Q_4(q)$ The neutral GQ $Q_4(q)$ of order (q, q) admitting the flag-transitive automorphism group isomorphic to the projective orthogonal group $PGO_4(q)$ extended by the field automorphism of order e.

$Q_3^+(q)$ The hyperbolic GQ $Q_3^+(q)$ of order $(q, 1)$ admitting the flag-transitive automorphism group isomorphic to the projective orthogonal group $PGO_4^+(q)$ extended by the field automorphism of order e.

$Q_5^-(q)$ The elliptic GQ $Q_4^-(q)$ of order (q, q^2) admitting the flag-transitive automorphism group isomorphic to the projective orthogonal group $PGO_6^-(q)$ extended by the field automorphism of order e.

We can verify that the duals of $W_3(q)$ and $H_3(q^2)$ are isomorphic to $Q_4(q)$ and $Q_5^-(q)$, respectively. Note that the dual of a GQ of order (s,t) is also a GQ of order (t,s). Sometimes the dual of $Q_3^+(q)$ (of order $(1,q)$) and the dual of $H_4(q^2)$ (of order (q^3,q^2)) are considered to be in a class of classical GQs.

2.6 Example (Sylvester quadrangle). This was found by J.J.Sylvester in 1844. We take a set Ω of letters $1,2,\ldots,6$, and define $\mathcal{G} = (\mathcal{P},\mathcal{L};*)$ by $\mathcal{P}$ = the transpositions on Ω, $\mathcal{L}$ = the 2^3-partitions on Ω, and $*$: symmetrized inclusion. The symmetric group S_6 on Ω acts on $\mathcal{G}$, which is transitive on the set of maximal flags. We can verify that $\mathcal{G}$ is a GQ of order $(2,2)$. Drawing a picture (or e.g.[Ca] Theorem 7.1.3), it is not so difficult to establish that there is a unique GQ of order $(2,2)$ up to isomorphism. Thus we have $S(6) \cong W(2)$, which also implies that $Aut(S(6)) = S_6 \cong Sp_4(2) = Aut(W(2))$.

Note that a geometry $\mathcal{P}_{2n+2}$ in 1.5 coincides with $\mathcal{S}_6$ if $n = 2$.

3. Tits' Characterization of Buildings.

3.1 An important observation. In §2, three classes of geometries of rank 2, that is, the generalized polygons, the circle geometries and the Petersen geometry, are introduced. The importance of these geometries lies in the following observation:

> Except Th and HN, each sporadic finite simple group acts flag-transitively on certain geometries in which every residues of flags of corank 2 are the generalized polygons, the circle geometries, or the Petersen geometry.

This fact was observed by many mathematicians including F. Buekenhout, A.A. Ivanov, W. Kantor, M. Ronan, S.D. Smith, G. Stroth and S. Shpectorov.

For example, in the projective space $\mathbf{PG}(n,q)$ in Example 1.2, each residue of cotype $\{i-1,i\}$ ($i = 1,\ldots,n-2$) is isomorphic to a projective plane $\mathbf{PG}(2,q)$, and hence a generalized 3-gon. Other residues of corank 2 are generalized digons.

In a classical polar space of rank r associated with a form (V,f) in Example 1.3, it is clear that residues of cotype $\{i,j\}$ with $|j-i| \geq 2$ are generalized digon. Since the residue of a maximal totally isotropic (or singular) subspace M is isomorphic to the projective space for M, any residue of cotype $\{i-1,i\}$ ($i = 1,\ldots,r-3$) is a generalized 3-gon. Take a flag X of cotype $\{r-2,r-1\}$, and let $X_{r-3} = W$ be the unique variety of $\mathcal{G}_{r-3}$ contained in X. Then $Res(X)$ consists of totally isotropic $(r-1)$- and r-subspaces containing W, which correspond to totally isotropic 1- and 2-subspaces in a vector space $W^\perp/W$ equipped with a non-degenerate form $\bar{f}$ inherited from f. Since they form a GQ for $(W^\perp/W,\bar{f})$, the residue $Res(X)$ is a generalized quadrangle.

In the geometry $\mathcal{S}_{2n+2}$ in Example 1.5, we take a flag X of cotype $\{n-2,n-1\}$, and let $X_{n-3} = M$ be the unique variety of $\mathcal{G}_{n-3}$ in X. As $|M| = 2(n-2)$, we may take $M = \{7,8,\ldots,2n+2\}$, and hence $Res(X)$ corresponds to a geometry consisting of pairs and partitions of type 2^3 of $\{1,\ldots,6\}$, which is a generalized quadrangle $\mathcal{S}_6 \cong W_3(2)$. It is clear that residues of cotype $\{i,j\}$ with $|i-j| \geq 2$ are generalized digons. The residue $Res(X)$

of a flag of cotype $\{i-1, i\}$ with $i = 1, \ldots, n-2$ corresponds to a geometry of points and pairs of a set of four points, which is the circle geometry on the four points.

3.2 Characterization via diagram. In view of the above observation, the following problem naturally occurs.

> Given an ordered set I and a family $\mathcal{D}(i,j)$ of geometries of rank 2 which is a certain family of generalized polygons, circle geometries or the Petersen graph for each $i, j \in I$ with $i < j$ (these detum are usually represented using a "diagram"), classify (flag-transitive) geometries $\mathcal{G}$ in which $Res(X)$ is isomorphic to a fixed geometry belonging to the specified family $\mathcal{D}(i,j)$ for every flag X of cotype $\{i,j\}$.

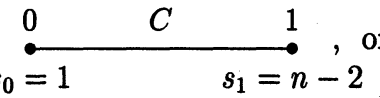
Much activity in the study of incidence geometry has been emanating from the attempt to solve this problem. This can be thought of as an attempt to chracterize a geometry in terms of its local structures only.

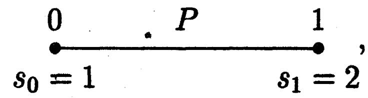
The above problem is known as a characterization via diagrams, because we usually use the "diagrams" in order to represent the local datum in a compact manner.

3.3 Diagrams. We say that a geometry $\mathcal{G}$ on I *admits a diagram* if for any $i, j \in I$ with $i < j$ the isomorphism type of the resdues $Res(X)$ for flags X of cotype $\{i,j\}$ depend only on i, j but not a particular choice of a flag. For example, flag-transitive geometry admits a diagram.

With a geometry on I admits diagram, we associate a diagram as follows: The diagram has nodes indexed by I and the nodes i and j ($i, j \in I$, $i < j$) are joined by the stroke with some symbol X showing the isomorphism class of the residues $Red_{c\mathcal{G}}(X)$ for all flags X of cotype $\{i,j\}$. Usually, we use the following convention to denote the isomorphism classes of geometries of rank 2.
(1) If the residue of cotype $\{i,j\}$ is a generalized n-gon for $n = 2, 3, 4$, we write

(2) If the residue of cotype $\{i,j\}$ is a circle geometry with n points or the Petersen geometry, we write (with orders)

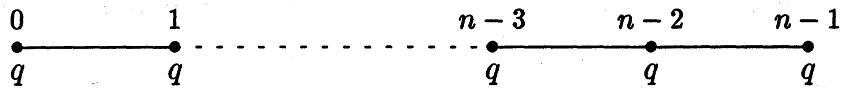


, or


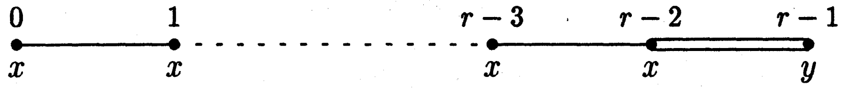
, respectively.

Of course, there are another classes of geometries of rank 2. However, they will not appear in my talk, so I will omit to introduce any convention to denote them. We also put the i-order s_i (see 1.1) under the node i ($\subset inI$). If we are given such a diagram describing the residual structure of a geometry $\mathcal{G}$, we say that a geometry $\mathcal{G}$ *belongs* to the diagram.

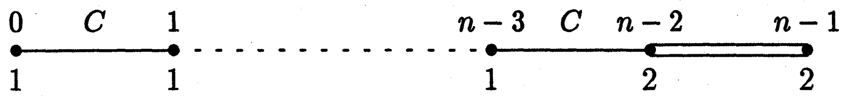
3.4 Examples. The projective geometry $\mathbf{PG}(n, q)$ belongs to the following diagram.



A finite classical polar space $\mathcal{G}$ of rank r belongs to the following diagrams, where $(x, y) = (q, q)$ if $\mathcal{G} = W_{2r-1}(q)$ or $Q_{2r}(q)$, $(x, y) = (q^2, q)$ if $\mathcal{G} = H_{2r-1}(q^2)$, $(x, y) = (q^2, q^3)$ if $\mathcal{G} = H_{2r}(q^2)$, $(x, y) = (q, 1)$ if $\mathcal{G} = Q^{+}_{2r-1}$, and $(x, y) = (q, q^2)$ if $\mathcal{G} = Q^{+}_{2r+1}$.



The geometry in Example 1.5 belongs to the following diagram. In [Me1], Thomas Meixner characterized the flag-transitive geometries of rank ≥ 4 belonging to this type of diagram (without specifying orders) modulo few cases.



In termes of the diagrams, the problem we posed in 3.2 can be expressed as follows:

> Given a diagram (with specified orders), determine all the (flag-transitive) geometries belonging to the diagram.

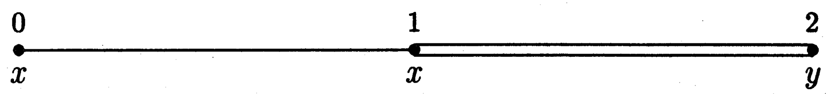
However, the given diagram (local datum) determines the most "universal" geometry with these local structures only, and the other geometries can be obtained as an epimorphic image of it. In order to make this idea clear, we need the notion of coverings.

3.5 Definition (Coverings). Let $\mathcal{G}$ and $\mathcal{H}$ be geometries on the same ordered set I. A map ρ from $\mathcal{G}$ to $\mathcal{H}$ is called a *covering* if it is a surjective map sending $\mathcal{G}_i$ onto $\mathcal{H}_i$ for each $i \in I$ such that the restriction of ρ to the residue $Res_{\mathcal{G}}(X)$ of each flag X of $\mathcal{G}$ of corank 2 gives an isomorphism from $Res_{\mathcal{G}}(X)$ onto $Res_{\mathcal{H}}(\rho(X))$. We also say that $\mathcal{G}$ *covers* $\mathcal{H}$ if there is a covering.

Tits is apparently one of the first mathematician to realize the importance of the characterization via diagrams and its relation with coverings. In his remarkable paper [Ti2], he studied geometries in which residues of corank 2 are generalized polygons and showed that they are covered by buildings, under some condition. Explicitly, he proved the following theorem:

3.6 Theorem. (Tits 1981 [Ti2] + Brouwer-Cohen 1983 [BC], see also [Ro] p.47 (4.9)) Assume that $\mathcal{G}$ is a geometry belonging to a diagram Δ which is a Coxeter diagram of type X_n. If Δ does *not* contain the C_3-diagram, then $\mathcal{G}$ can be obtained as a quotient of a building of type X_n.

3.7 Classification of C_3-geometries. The exceptional case of rank 3 is a geometry belonging to the following diagram, the diagram of type C_3. We will call any geometry belonging to this diagram a C_3-geometry.



Since all thick buildings belonging to Coxeter diagrams of spherical type of rank ≥ 3 were classified by Tits in 1974 [Ti1], the above theorem 3.6 allows us to obtain quite explicit information on a geometry by simply analyzing its local structures. The above hypothesis on C_3-diagram is essential, because the sporadic A_7-geometry described below is a C_3-geometry which is not a building. This is why characterization and classification of C_3-geometry is recognized as one of the main problems in the study of diagram geometry.

I will survey some results on classification of (flag-transitive) C_3-geometries with finite orders and their circular extensions. Before that, let me introduce an exceptional C_3-geometry.

3.8 The Sporadic A_7-geometry. Here I describe an exceptional C_3-geometry (see [Ro] p.50 or [Ca] p.89–90.) This geometry $\mathcal{G} = (\mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_2; *)$ is defined as follows: First, we set $\mathcal{G}_0 :=$ the 7 letters of $\Omega = \{1, 2, \ldots, 7\}$ and $\mathcal{G}_1 :=$ the 35 (unordered) triples of of Ω. We consider a projective plane having Ω as the set of points. Such plane should be of order 2 and can be determined by specifying its 7 lines. For example, $\Pi = (\Omega, \mathcal{L})$ is a projective plane, where $\mathcal{L}$ consists of the lines 123, 145, 167, 246, 257, 347 and 356. Here we also denote a line by the triple of points on it. It can be verified that there are 30 such planes, which form two orbits of the same length 15 under the action of the alternating group A_7 on Ω. Two planes belong to the same A_7-orbit if and only if they have exactly one line in common. Now we define $\mathcal{G}_2$ as one of these two A_7-orbits, and determine $*$ by natural containment.

The resulting geometry is a C_3-geometry, which called the *sporadic A_7-geometry*. For the residues of lines and planes, it is immediate to see their structures. For a point, say 7, the set of lines incident with 7 can be identified with the set of permutaions on $\{1, \ldots, 6\}$. A plane incident with 7 can be determined by its three lines through 7, which correponds to a permutation of type 2^3 on $\{1, \ldots, 6\}$. Thus the residue at 7 is isomorphic to the GQ $S(6)$ in 2.6.

4. Results on flag-transitive C_3-geometry.

There are several results on characterizing C_3-geometry. In this section, we introduce some of them. In the following, $\mathcal{G}$ will denote a C_3-geometry. I recommend to the readers the nice survey by Lunardon and Pasini [LP2] for more detailed information about the results on C_n-geometries up to 1990.

4.1 Theorem. (Tits 1981 [Ti2], For an elementary proof, see Pasini's textbook [Pa3] Cor.7.39) If there is at most one line incident with two distinct points, then $\mathcal{G}$ is a building.

Note that the sporadic A_7-geometry does not satisfy the assumption of this theorem, because there are 5 lines $12i$ ($i = 3, \dots, 7$) through two points 1 and 2.

Non-thick C_3-geometries are characterized by [Re] as quotients of Klein quadrics. Thus from now on we will assume that $\mathcal{G}$ is thick.

It can be verified that the diameter of the collinearity graph of $\mathcal{G}$ is at most 2 [Pal] p.50 Cor.1. Thus, the locally finiteness implies the finiteness of $\mathcal{G}$. Furthermore, we can obtain exact formulas on the number of points, lines and planes, in terms of x, y and the following important constant.

4.2 Ott-Liebler number. First we can verify that two distinct planes u and v are incident with at most one line in common. If $\mathcal{G}_1(u) \cap \mathcal{G}_1(v) \neq \emptyset$, we say that u and v are *cocolinear* and denote by $u \cap v$ the unique line incident with u and v. Whenever we use the notation $u \cap v$, we assume that u is cocolinear with v ($\neq u$). Now for a point-plane flag (a, u), we set $\alpha(a, u) := \{v \in \mathcal{G}_2(a) | a \not* (u \cap v)\}$.

We can prove that this number does not depend on the particular choice of a point-plane flag [Pa2] Theorem 1. The constant $\alpha(a, u)$ will be called the *Ott-Liebler numebr* and denoted by α. Ott and Liebler indipendently tried to analize the multiplicities of the irreducible representations of the Hecke algebra obtained from $\mathcal{G}$. They first noticed that these multiplicities are described in terms of x, y and α, but in their work, $\alpha = \alpha(P, u)$ is interpreted as the number of closed galleries of type 012012012 based at a maximal flag (P, l, u). They derived many divisibility conditions among x, y, α together with the well-definedness of α. The more geometric definition above is due to Pasini, which enables us to verify some results of Ott and Liebler by elementary counting arguments.

Note that if $\mathcal{G}$ is a building, then $\alpha = 0$. Conversely, we may verify that if $\alpha = 0$ then the hypothesis of Theorem 3.6 is satisfied, and therefore $\mathcal{G}$ is a building. For the sporadic A_7-geometry, we have $\alpha = \alpha(1, \pi) = \#$(lines l of π not through 1) $\#$(planes through l distinct from π)$= 4 \cdot 2 = 8$.

4.3 Theorem. (Pasini 1986 [Pa2]4(1)(3)) If $\mathcal{G}$ is locally finite of order (x, y) with the Ott-Liebler number α, $\mathcal{G}$ has $|\mathcal{G}_0| = (x^2 + x + 1)(x^2y + 1)/(\alpha + 1)$ points,
$|\mathcal{G}_1| = (x^2 + x + 1)(x^2y + 1)(xy + 1)/(\alpha + 1)$ lines and
$|\mathcal{G}_2| = (x^2y + 1)(xy + 1)(y + 1)/(\alpha + 1)$ planes. Furthermore, x divides α.

4.4 Flag-transitivity. So far we do not assume anything on the full automorphism group of $\mathcal{G}$. In fact, every examples so far we met are both locally finite (except projective spaces over an infinite division rings) and flag-transitive. Motivated by exceptional behavior of small Lie type groups, Aschbacher and Steve Smith rediscovered the sporadic A_7-geometry in 1980. Aschbacher also tried to characterize this geometry by its flag-transitivity:

4.5 Theorem. (Aschbacher 1984 [As]) Let $\mathcal{G}$ be a locally finite, thick, flag-transitive C_3-geometry. If $Res(u)$ for a plane u is a Desarguesian projective plane (see 2.2) and $Res(a)$ for a point a is a thick classical GQ (see 2.3), then $\mathcal{G}$ is a building or the sporadic A_7-geometry.

We will call a C_3-geometry *anomalous* if it is not a building nor the sporadic A_7-geometry. We are now in the position to state the following remarkable conjecture.

4.6 Conjecture. There is no locally finite, thick, flag-transitive anomalous C_3-geometry.

Pasini and Lunardon proved some results generalizing Theorem 3.6, but the complete solution of the conjecture has not yet been obtained.

4.7 Theorem. If $\mathcal{G}$ is a flag-transitive, locally finite, thick, anomalous C_3-geometry, we have
(1) (see [LP2] Prop.12) The residue of a plane is non-Desarguesian, and
(2) (Lunardon and Pasini [LP1]) $\mathcal{G}$ is not flat.

Recently, Antonio Pasini and I made a new contribution to the solution of the conjecture [YP]. The result, in a sense, solved the conjecture in over the three quarter of the possible cases.

4.8. Theorem. (Yoshiara and Pasini 1993 [YP]) If $\mathcal{G}$ is locally finite, flag-transitive and anomalous, then $Aut(\mathcal{G})$ is non-solvable and y is odd. (x should be even).

It seems unlikely that there exists a generalized quadrangle of order (s,t) with $s-t$ odd. Thus this result forces very restrictive conditions on the structure of the point-residue of $\mathcal{G}$, which is a GQ of order (x,y). The author and Antonio Pasini hope the remaining case will be eliminated in near future. The rough outline of the proof is described in [Yo3], so I will not repeat it.

5. The $C_3.c^*$-geometries.

I was involved in the study of C_3-geometry during my classification program of flag-transitive C_nc^*-geometries, so called, the *extended dual polar spaces.* As for the motivations of this program and the related results, see the author's survey [Yo2], the paper [Yol] and the note [PY2].

5.1 Definition. A geometry $\mathcal{G} = (\mathcal{G}_0, \ldots, \mathcal{G}_{r-1})$ is called a *circular extension*) of a geometry $\mathcal{H} = (\mathcal{H}_0, \ldots, \mathcal{H}_{r-2})$ if each residue in $\mathcal{G}$ of cotype $\{1, \ldots, r-1\}$ is isomorphic to $\mathcal{H}$ and a residue of cotype $\{0, i\}$ is a circle geometry for $i = 1$ and a generalized digon for $i = 2, \ldots, r-1$. In this case, we say that $\mathcal{G}$ is a *$c.\mathcal{H}$-geometry.* The dual of a $c.\mathcal{H}$-geometry is called a *dual circular extension* of a geometry $\mathcal{K} = (\mathcal{H})^*$, the dual of $\mathcal{H}$, or a $(\mathcal{K}).c^*$-geometry

As a corollary of Theorem 4.8 above, we proved that

5.2. Theorem. (Yoshiara and Pasini 1993 [YP]) There is no flag-transitive C_3c^*-geometry $\mathcal{G}$ in which a residue of cotype $\{0,1,2\}$ is an anomalous C_3-geometry.

If such geometry exists, we can show that the stabilizer G_P of a point P in the full automorphism group G of $\mathcal{G}$ is solvable. Thus we may apply Theorem 5.1.

By this results, if we are interestead in classifying flag-transitive $C_3.c^*$-geometries, we may assume that the residues of cotype $\{0,1,2\}$ is either a classical polar space or the sporadic A_7-geometry.

5.3 $C_3.c^*$-geometry with classical C_3-residues. This is one of the most interesting geometries which has not yet been completely classified. The possible polar spaces as the residues are explicitly determined in [Yo1]. In the table bellow, I indicate the current situation of the classification, where, instead of describing examples in geometric terms, some flag-transitive groups are given. In the second and the third columns, the isomorphism type of residues of cotype $\{0,1,2\}$ and the *dual* of the residues of cotype $\{1,2,3\}$ are given. The residues of cotype $\{1,2,3\}$ are flag-transitive $C_2.c^*$-geometries in which the residues of cotype $\{2,3\}$ are classical generalized quadrangles. Such geometries are completely classified (see e.g. [PY1]). I use the notation in [Yo2] to denote the isomorphism classes of these geometries.

Note that classifications in some cases are now completed by a result of Sasha Ivanov [Iv], which was established after the conference.

	C_3-residue	$c.C_2$-residue	Known Examples	Classified?
(1)	$W_5(2)$	$\mathcal{A}_\infty$	$2(2^6 \times 2_+^{1+8})S_6(2)$, $S_8(2)$	yes [Yo1]
(2)	$W_5(2)$	$\mathcal{A}_+$	?	Not yet but see [Yo1]
(3)	$W_5(2)$	$\mathcal{A}_-$	$3.F_{22}$	yes [Iv]
(4)	$H_5(4)$	$\mathcal{K}^+$	$U_6(2)$	yes [Yo1]
(5)	$H_5(4)$	$\mathcal{K}^-$	$Co_2 \times 2$	yes [Yo1] see also [Me2]
(6)	$Q_7^-(2)$	$\mathcal{O}$	F_{24}	Not yet but see [Iv]
(7)	$Q_7^-(2)$	$\bar{\mathcal{O}}$	?	Not yet
(8)	$Q_6(3)$	$\mathcal{U}$	F_{24}	Not yet
(9)	$Q_7^-(3)$	$\mathcal{S}$	**M**	Not yet

When a C_3-residue is the sporadic A_7-geometry, the classification is rather easy. In [PY2], the following theorem was proved.

5.6 Theorem. (1) There is a unique flag-transitive $C_3.c^*$- geometry $\mathcal{G}$ with C_3-residues isomorphic to the sporadic A_7-geometry.
(2) There is a unique flag-transitive $c.C_3$- geometry $\mathcal{G}$ with C_3-residues isomorphic to the sporadic A_7-geometry.
(3) There is a unique flag-transitive $c.C_3.c^*$- geometry $\mathcal{G}$ with C_3-residues isomorphic to the sporadic A_7-geometry.
(4) There is no flag-transitive $c.(c.C_3.c^*)$- nor $(c.C_3.c^*).c^*$- geometry $\mathcal{G}$ with C_3-residues isomorphic to the sporadic A_7-geometry.

I will conclude my talk with a description of the above $c.C_3.c^*$-geometry.

5.5 A geometry related to $S(24,8,5)$. Let $(\Omega, \mathcal{B})$ be the Steiner system $S(24,8,5)$, that is, Ω is a set of 24 letters and $\mathcal{B}$ is a family of 8-subsets of Ω with the property that for each 5-subset F of Ω there is a unique element $C \in \mathcal{B}$ containing F. Elements of $\mathcal{B}$ are called *octads*. The automorphism group of $(\Omega, \mathcal{B})$ (that is, the group of permutations on Ω preserving $\mathcal{B}$) is the *Mathieu group* $M = M_{24}$ of degree 24.

We now fix an octad C and define a geometry $\mathcal{G}$ over $\{0, \ldots, 4\}$ as follows (This definition is slightly simpler than that given in [PY1], where a PLANE is defined as $(C \cap D, (\Omega - C) \cap D)$ for a "plane" in the sense given bellow): The sets $\mathcal{G}_0$ of *points* and $\mathcal{G}_1$ of *lines* are the set of 8 letters in C and the set of $\binom{8}{2} = 28$ 2-subsets of C, respectively. Dually, the sets of $\mathcal{G}_4$ of *dual points* and $\mathcal{G}_3$ of *dual lines* as the set of 16 letters in $\Omega - C$ and the set of $\binom{16}{2} = 120$ 2-subsets of $\Omega - C$, respectively. The set $\mathcal{G}_2$ of *planes* is defined to be the set of octads D with $|C \cap D| = 4$.

Every varieties of $\mathcal{G}_0 \cup \mathcal{G}_1$ are incident to all varieties of $\mathcal{G}_3 \cup \mathcal{G}_4$, and the incidence on $\mathcal{G}_0 \cup \mathcal{G}_1 \cup \mathcal{G}_2$ or $\mathcal{G}_2 \cup \mathcal{G}_3 \cup \mathcal{G}_4$ is given by natural inclusion.

The resulting geometry $\mathcal{G} = (\mathcal{G}_0, \ldots, \mathcal{G}_4)$ of rank 5 turns out to be a flag-transitive $c.C_3.c^*$-geometry with the automorphism group $2^4 : A_8$, the stabilizer of an octad C in M_{24}, in which the residues of cotype $\{0,4\}$ are isomorphic to the sporadic A_7-geometry. By Theorem 5.6, there is a unique such geometry up to isomorphism and other geometries of type $c.C_3$ and $C_3.c^*$ stated in the theorem can be obtained as the residues of this geometry.

5.6 Verification. Here I will verify that any residue of cotype $\{0,4\}$ in the geometry $\mathcal{G}$ given in 5.5. is, in fact, the sporadic A_7-geometry (see 3.8). The other residues are easy to observe.

First, recall some properties of the Steiner system $S(24,8,5)$. Any two distinct octads interset in exactly 0, 2 or 4 letters. For every 4-subset T of C, there are exactly 4 octads D distinct from C with $C \cap D = T$. This implies that the set $\mathcal{G}_2 = \{D \in \mathcal{B} \| C \cap D| = 4\}$ of planes of our geometry consists of $\binom{8}{4} \cdot 4 = 280$ octads. The stabilizer $G = M_C$ of an octad in $M = Aut((\Omega, \mathcal{B})) \cong M_{24}$ induces A_8 on the 8 letters in C with the kernel $K \cong 2^4$, while G acts faithfully and doubly transitively on $\Omega - C$ with regular normal subgroup K. The stabilizer $G_{P'}$ of a letter $P' \in (\Omega - C)$ acts faithfully on C as A_8. In particular, G acts transitively on the set of pairs (P, P') of letters $P \in C$ and $P' \in (\Omega - C)$, that is, a flags of type $\{0,4\}$.

We can now fix a flag (P, P') ($P \in \mathcal{G}_0$, $P' \in \mathcal{G}_4$) of type $\{0,4\}$. The set $\mathcal{G}_1(P, P')$ of lines incident to (P, P') consists of 7 lines bijectively corresponding to the letters in $\Sigma := C - \{P\}$. The set $\mathcal{G}_2(P, P')$ consists of octads D with $|C \cap D| = 4$ containing P and P'. The map $D \mapsto (D \cap \Sigma)$ gives a bijection from $\mathcal{G}_2(P, P')$ to the set of triples of letters in Σ, because there is a unique octad D containing a 5-subset $T \cup \{P, P'\}$ for each 3-subset T of Σ and this octad D intersects C in eaxctly $T \cup \{P\}$.

Consider the set $\mathcal{G}_3(P, P')$. For each dual line $l' = \{P', Q'\}$ of $\mathcal{G}_3(P, P')$, where Q' is a letter in $\Omega - C$, we will observe that $\mathcal{G}_2(P, l', P')$ can be identified with a set of 7 "projective lines" affording a structure of the projective plane $\mathbf{PG}(2,2)$ on the set Σ of 7 letters. To see this, it suffices to show that there is a unique octad $D \in \mathcal{G}_2(P, l', P')$ containing two distinct letters Q, R of Σ and that there is a unique letter $Q \in \Sigma$ with $(\Sigma \cap D) \cap (\Sigma \cap D') = \{Q\}$ for any two distinct octads D and D' of $\mathcal{G}_2(P, l', P')$. Since the unique octad containing the 5-subset $\{P, Q, R, P', Q'\}$ intersects C in exactly 4 letters, the first condition above is satisfied.

Let D and D' be distinct octads of $\mathcal{G}_2(P, l', P')$. Since $D \cap D'$ contains P, P', Q', we have $D \cap D' = \{P, P', Q', Q\}$ for some letter Q. In particular, the symmetric difference $D \oplus D' = (D - D') \cup (D' - D)$ is an octad. If $Q \notin C$, we have $C \cap D \cap D' = \{P\}$. Then $|C \cap (D - D')| = 3 = |C \cap (D' - D)|$, as $|C \cap D| = |C \cap D'| = 4$. However, this implies two octads C and $D \oplus D'$ intersect in exactly 6 letters, a contradiction. Thus 3-subsets $(D \cap \Sigma)$ and $(D' \cap \Sigma)$ of Σ intersect in exactly one letter Q, which proves the second claim above.

Thus we may identify the set $\mathcal{G}_2(P, l', P')$ with a set of 7 "projective lines" on Σ. Since $G_{P'} \cong A_8$, the stabilizer $G_{P,P'}$ of the letters $P \in C$ and $P' \in (\Omega - C)$ are isomorphic to A_7. We can verify that it acts transitively on the 15 non-zero vectors of the normal subgroup $K \cong 2^4$ of G (which is an exceptional phenomenon). Then $G_{P,P'}$ is transitive on the 15 letters $\Omega - C - \{P'\}$, which bijectively corresponds to $\mathcal{G}_3(P, P')$. Thus $\mathcal{G}_3(P, P')$ forms an orbit under the action of A_7, and hence it coincides with the set of PLANES of the sporadic A_7-geometry (see 3.8).

We observed that $\mathcal{G}_1(P, P')$, $\mathcal{G}_2(P, P')$ and $\mathcal{G}_3(P, P')$ are bijectively correspond to the sets of POINTS, LINES and PLANES of the sporadic A_7-geometry defined in 3.8, respectively. The incidence of $Res_{\mathcal{G}}((P, P'))$ inherited from $\mathcal{G}$ coincides with the (natural) incidence in the sporadic A_7-geometry, and therefore $Res_{\mathcal{G}}((P, P'))$ is isomorphic to the sporadic A_7-geometry.

Since $G_{P,P'} \cong A_7$, we can also conclude that $G_{P,P'}$ acts flag-transitively on $Res((P, P'))$, and hence $\mathcal{G}$ admits a flag-transiitve group $G \cong 2^4 : A_8$.

References

[As] M. Aschbacher, Finite geometries of type C_3 with flag-transitive automorphism groups, *Geom. Dedicata* **16** (1984) 195–200.

[Bu] F. Buekenhout, The basic diagram of the geometry, pp.1–29, in *Geometries and Groups*, Lecture Notes in Math. **893**, Springer, 1981.

[BC] A. Brouwer and A. Cohen, Some remarks on Tits's geometries, *Indag. Math.* **45** (1983) 393–402.

[Ca] P.J. Cameron, *Projective and Polar Spaces*, QMW Math Notes 13, University of London, 199?.

[Iv] A.A. Ivanov, On geometries of the Fischer groups, preprint (February 14, 1994).

[Ka] W. Kantor, Primitive groups of odd degree and an application to finite projective planes, *J. Algebra* **106** (1987) 14–45.

[LP1] G. Lunardon and A. Pasini, A result on C_3-geometries, *European J. Combin.* **10** (1989) 265–271.

[LP2] G. Lunardon and A. Pasini, Finite C_n geoemtries: a survey, *Note Math.* **10** (1990) 1–35.

[Me1] T. Meixner, Some polar towers, *Europ. J. Combin.* **12** (1991) 397–451.

[Me2] T. Meixner, A geometric characterization of the simple group $Co2$, *to appear.*

[Pa1] A. Pasini, On certain geometries of type C_n and F_4,*Discrete Math.* **58** (1986) 45–61.

[Pa2] A. Pasini, On geometries of type C_3 that are either buildings or flat, *Bull. Soc. Math. Belg. B* **38** (1986) 75–99.

[Pa3] A. Pasini, *An Introduction to Diagram Geometry*, Oxford University Press, to be published.

[Pa4] A. Pasini, On finite geometries of type C_3, *Note Math.* **6** (1986) 205–236.

[PY1] A. Pasini and S. Yoshiara, Flag-transitive Buekenhout geometries, pp. 403–447 in "Combinatorics'90", *Annals Disc. Math.* **52**, North Holland, 1992.

[PY2] A. Pasini and S. Yoshiara, Generalized towers of flag-transitive circular extensions of a non-classical C_3-geometry, to appear in *J. Combin. Th. A.*

[Re] S. Rees, A classification of a class of C_3 geometries, *J. Combin. Th. A* **44** (1987) 173–181.

[Ro] M. Ronan, *Lectures on Buildings*, Perspectives in Math. **7**, Academic Press, London, 1989.

[Ti1] J. Tits, *Buildings of Spherical type and Finite BN-pairs*, Lecture Notes in Math. **386**, Springer, 1974.

[Ti2] J.Tits, A local approach to buildings, pp.519–547 in *The Geometric Vein*, Springer, 1981.

[Yo1] S. Yoshiara, On some extended dual polar spaces I, *European J. Combin.* **15** (1994) 73–86.

[Yo2] S. Yoshiara, On flag-transitive $C_3.c^*$-geometries, in the reports of "the conference on Association Schemes", held at Kyoto University, March, 1993.

[Yo3] S. Yoshiara, in the reports of "The 11th Conference of Algebraic Combinatorics", held at Osaka Kyoiku University, June, 1993.

[YP] S. Yoshiara and A. Pasini, On flag-transitive anomalous C_3-geometries, *Contributions to Algebra and Geometry* **34** (1993) 277–286.